

ON THE GENERALIZED ZAKHAROV-KUZNETSOV EQUATION AT CRITICAL REGULARITY

AXEL GRÜNROCK

ABSTRACT. The Cauchy problem for the generalized Zakharov-Kuznetsov equation

$$\partial_t u + \partial_x \Delta u = \partial_x u^{k+1}, \quad u(0) = u_0$$

is considered in space dimensions $n = 2$ and $n = 3$ for integer exponents $k \geq 3$. For data $u_0 \in \dot{B}_{2,q}^{s_c}$, where $1 \leq q \leq \infty$ and $s_c = \frac{n}{2} - \frac{2}{k}$ is the critical Sobolev regularity, it is shown, that this problem is locally well-posed and globally well-posed, if the data are sufficiently small. The proof follows ideas of Kenig, Ponce, and Vega [14] and uses estimates for the corresponding linear equation, such as local smoothing effect, Strichartz estimates, and maximal function inequalities. These are inserted into the framework of the function spaces U^p and V^p introduced by Koch and Tataru [17], [18].

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1. INTRODUCTION

The Zakharov-Kuznetsov equation (ZK)

$$(1) \quad \partial_t u + \partial_x \Delta u = \partial_x u^2$$

with $\Delta = \partial_x^2 + \sum_{i=1}^{n-1} \partial_{y_i}^2$, $(x, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$, is a generalization of the famous Korteweg-de Vries equation (KdV) to arbitrary higher dimensions. In 1974, Zakharov and Kuznetsov derived (1) as a model describing the unidirectional wave propagation in a magnetized plasma in three space dimensions [33, equation (6)]. For two dimensions, a derivation of (1) from the basic hydrodynamic equations is due to Laedke and Spatschek [19, Appendix B]. We also refer to the paper [20] by

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Lannes, Linares, and Saut for a rigorous justification of ZK valid for $n \in \{2, 3\}$. Both, the Cauchy problem as well as several initial boundary value problems connected with (1) have attracted considerable interest in recent years, we mention [2], [3], [5], [11], [24], [26], [28]; [6], [21], [25], [30]. This list is by no means exhaustive. Similar as for KdV and - to the author's knowledge - beginning with the work [2] of Biagioni and Linares on the modified equation, generalizations of ZK with higher power nonlinearities

$$(2) \quad \partial_t u + \partial_x \Delta u = \partial_x u^{k+1} \quad \text{with} \quad u(0) = u_0$$

are considered, too. We call (2) the k -th generalized ZK equation, for short gZK- k . In this paper we are concerned with local and small data global well-posedness of the Cauchy problem for gZK- k in two and three space dimensions for integers $k \geq 3$. (Unfortunately our arguments break down for the modified equation, i. e. for $k = 2$.) In the 2 D - case the following results are known for data in the classical Soboles spaces H^s .

- In 2011 Linares and Pastor [23] showed that the Cauchy problem for gZK- k is locally well-posed in H^s , if $k \geq 2$ and $s > \max(\frac{3}{4}, 1 - \frac{3}{2k-4})$. If the data are sufficiently small in H^1 , then the corresponding solutions extend globally in time.
- For $k > 8$ the lower bound on s was pushed down to $s > 1 - \frac{2}{k}$ by Farah, Linares, and Pastor [7] in 2012. Since $s_c = 1 - \frac{2}{k}$ is the critical regularity by scaling considerations, this result covers the whole subcritical range.
- Further progress on the local problem was reached by Ribaud and Vento [29] in 2012. Their results almost reached s_c for all $k \geq 4$, while for the quartic nonlinearity they assume $s > \frac{5}{12}$.

Further results on gZK- k in two dimensions with data in weighted spaces were recently obtained by Fonseca and Pachon [8]. The author is not aware of any comparable results for $k \geq 3$ in the three dimensional case, where the critical regularity is $s_c = \frac{3}{2} - \frac{2}{k}$. More generally we have

$$s_c = s_c(n, k) = \frac{n}{2} - \frac{2}{k}.$$

Roughly speaking, the method of proof is the same in all three papers [23], [7], and [29]. The authors adapt the strategy developed by Kenig, Ponce, and Vega in [14] in the KdV-context and apply a combination of local smoothing estimate, Strichartz inequality, and maximal function estimate in a contraction mapping argument. Here we shall pick up these ideas, push them down to the critical regularity and extend the arguments to the three dimensional case. Following Molinet-Ribaud [27] and especially in our method of proof Koch-Marzuola [16] in their works on gKdV, we consider data in the homogeneous Besov spaces

$$\dot{B}_{2,q}^s = \{u_0 \in \mathcal{Z}' : \|u_0\|_{\dot{B}_{2,q}^s}\},$$

where \mathcal{Z}' is the dual space of

$$\mathcal{Z} = \{f \in \mathcal{S} : (D^\alpha \mathcal{F}f)(0) = 0 \text{ for every multi-index } \alpha\}.$$

Here and below \mathcal{F} denotes the Fourier transform. For $q < \infty$ the Besov-norm is in general given by

$$\|u_0\|_{\dot{B}_{p,q}^s} = \left(\sum_{N \in 2^{\mathbb{Z}}} \|P_N u_0\|_{L^p} \right)^{\frac{1}{q}},$$

$P_N = \mathcal{F}^{-1} \chi_{\{|\xi| \sim N\}} \mathcal{F}$ are the Littlewood-Paley projections. A case of special interest is $q = 2$, where $\dot{B}_{2,2}^s = \dot{H}^s$, the homogeneous Sobolev (or Riesz-potential) space.

For $q = \infty$ one has the usual modification $\|u_0\|_{\dot{B}_{p,\infty}^s} = \sup_{N \in 2^{\mathbb{Z}}} \|P_N u_0\|_{L^p}$, and in this case (with $p = 2$) we will in addition assume for our data, that

$$\lim_{N \rightarrow \infty} \|P_N u_0\|_{L^2} = \lim_{N \rightarrow 0} \|P_N u_0\|_{L^2} = 0.$$

With $\dot{B}_{2,\infty}^{s,o}$ we will denote the closed subspace of all $u_0 \in \dot{B}_{2,\infty}^s$, for which these limits vanish. Then \mathcal{Z} is dense in $\dot{B}_{2,\infty}^{s,o}$. Without this additional assumption, several of our arguments break down, e. g. we loose the persistence property of the solution. Observe for $1 \leq \tilde{q} \leq 2 \leq q < \infty$ the inclusions

$$\dot{B}_{2,1}^s \subset \dot{B}_{2,\tilde{q}}^s \subset \dot{H}^s \subset \dot{B}_{2,q}^s \subset \dot{B}_{2,\infty}^{s,o} \subset \dot{B}_{2,\infty}^s,$$

so for fixed s_c , on the fine scale of the q 's, the $\dot{B}_{2,\infty}^{s_c,o}$ is the largest data space we cope with. After these preparations we can state our

Main result: *Let $n \in \{2, 3\}$ and $k \geq 3$ an integer. Then the Cauchy problem (2) is locally well-posed for data in $\dot{B}_{2,q}^{s_c}$, if $q < \infty$, and in $\dot{B}_{2,\infty}^{s_c,o}$. Moreover, we have global well-posedness for small data in these spaces.*

A more precise statement will follow at the end of Section 2. We remark already, that no smallness assumption is needed for the local part, but that - as usual in a critical case - the lifespan of the solutions cannot be controlled by the size of the data in their natural norm. To obtain the result, two main difficulties have to be overcome. The first is to prove a sharp global maximal function estimate or to find a substitute for this. In 2 D we can solve this problem by symmetrizing the equation, see Section 3.1, especially Proposition 1 below, while in 3 D a surprisingly soft argument allows us to circumvent this obstacle, see Section 5.1. The second problem is the missing generalized Leibniz rule in higher dimensional mixed Lebesgue spaces of type $L_x^p L_t^q$. This is solved by using the spaces $U^p(L_x^2)$ and $V^p(L_x^2)$ of L_x^2 -valued functions of the time variable, which were introduced by Koch and Tataru in [17], [18], see also the exposition by Hadac, Herr, and Koch in [12] and Koch's lecture [15]. Since the norms of these spaces depend on the size of the spatial Fourier transform, the "distribution" of derivatives on various factors can easily be handled. Some basics about these spaces, as far as needed here, are gathered in Section 2.

In proving the result, we can restrict ourselves to apply *linear* estimates for free solutions - no bilinear refinement of a Strichartz type inequality is used. For $k = 3$ this is astonishing, if we compare our results here with the theory for gKdV. Using linear estimates only, Kenig, Ponce, and Vega obtained well-posedness for gKdV-3 in $H^s(\mathbb{R})$ for $s \geq \frac{1}{12}$. To push this down to the critical regularity, a bilinear estimate for free solutions is needed, see the result in [9] by the author, which was later on improved by Tao [31] to the endpoint and by Koch-Marzuola [16] to critical Besov spaces. As our calculations show, linear estimates are sufficient in higher dimensions. Furthermore we remark that for the quartic nonlinearity in 2 D our result closes a gap of $\frac{1}{12}$ derivatives between the existing LWP theory and the scaling heuristic.

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2. FUNCTION SPACES AND PRECISE STATEMENT OF RESULTS

Here we collect the necessary facts about the function spaces U^p and V^p , respectively U_φ^p and V_φ^p . For proofs and detailed descriptions we refer to the works [12] and [15]. We begin with the functions of bounded p -variation, which were (in the real valued case) introduced by Wiener in [32]. Let $I \subset \mathbb{R}$ be an interval and \mathcal{P}_I denote the system of all finite partitions $P = \{t_0 < \dots < t_K\} \subset I$ of I . Here $t_K = \infty$ is admitted, if I is unbounded to the right. For a function $v : I \rightarrow L_x^2$ the p -variation $\omega_p(I, v)$ is defined by

$$\omega_p(I, v) := \sup_{P \in \mathcal{P}_I} \left(\sum_{k=1}^K \|v(t_k) - v(t_{k-1})\|_{L_x^2} \right)^{\frac{1}{p}},$$

which, for $1 \leq p < \infty$, is a seminorm. Setting

$$\|v\|_{V^p} := \max(\|v\|_{L_I^\infty(L_x^2)}, \omega_p(I, v))$$

we get a norm on the linear space

$$V^p(L_x^2) := \{v : I \rightarrow L_x^2 : \omega_p(I, v) < \infty\},$$

which thereby becomes a B-space. Functions in $V^p(L_x^2)$ are not necessarily continuous, but one sided limits always exist. The closed subspace of all right continuous functions in $V^p(L_x^2)$ is denoted by $V_{rc}^p(L_x^2)$. For $1 \leq p < q < \infty$ the embeddings

$$V^p(L_x^2) \subset V^q(L_x^2) \subset L_I^\infty(L_x^2)$$

are continuous. Closely related are the function spaces $U^p(L_x^2)$, where again $1 \leq p < \infty$. Let $P = \{t_0 < \dots < t_K\}$ be a partition as above and $\psi_1, \dots, \psi_K \in L_x^2$. Then the step function

$$a = \sum_{k=1}^K \chi_{[t_{k-1}, t_k)} \psi_k$$

is called a U^p -atom, if $\sum_{k=1}^K \|\psi_k\|_{L_x^2}^p = 1$. One says that $u \in U^p(L_x^2)$, if there exist sequences $(\lambda_j)_{j \in \mathbb{N}} \in \ell^1(\mathbb{N})$ and $(a_j)_{j \in \mathbb{N}}$ of U^p -atoms, so that $u = \sum_{j=1}^\infty \lambda_j a_j$. These functions constitute a linear space, which endowed with the norm

$$\|u\|_{U^p} := \inf \left\{ \sum_{j=1}^\infty |\lambda_j| : u = \sum_{j=1}^\infty \lambda_j a_j \right\}$$

becomes a B-space. If $1 \leq p < q < \infty$ the embeddings

$$U^p(L_x^2) \subset U^q(L_x^2) \subset L_I^\infty(L_x^2)$$

are continuous. U^p -functions are continuous from the right. These two scales of function spaces are tied by continuous embeddings. Assume once more $1 \leq p < q < \infty$. Then we have

$$(3) \quad U^p(L_x^2) \subset V^p(L_x^2) \quad \text{and} \quad V_{rc}^p(L_x^2) \subset U^q(L_x^2).$$

Comparing with Besov-norms (of L_x^2 -valued functions) we have the inequalities

$$\|v\|_{\dot{B}_{p,\infty}^{\frac{1}{p}}} \lesssim \|v\|_{V^p} \quad \text{and} \quad \|u\|_{U^p} \lesssim \|u\|_{\dot{B}_{p,1}^{\frac{1}{p}}}.$$

Apart from the embeddings above, the U^p 's and V^p 's are connected by duality. In fact for $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$ we can identify

$$(4) \quad (U^p(L_x^2))' \simeq V^{p'}(L_x^2),$$

where the dual pairing $B : U^p(L_x^2) \times V^{p'}(L_x^2) \rightarrow \mathbb{C}$ is given by a generalized Stieltjes integral

$$B(u, v) = \int_I \langle u, dv \rangle = \int_I \langle u, \frac{\partial v}{\partial t} \rangle dt,$$

the latter, if v has a locally integrable weak derivative. We refer to Sections 3.5, 3.8, and 3.10 of [15] for details on the duality between U^p and $V^{p'}$.

Now Bourgain's construction of the $X^{s,b}$ -spaces is repeated. Let $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a phase function and $(U_\varphi(t))_{t \in \mathbb{R}}$ the unitary group associated with the linear equation $u_t = i\varphi(-i\nabla)u$. Then one defines

$$V_\varphi^p = U_\varphi V^p(L_x^2) \quad \text{and} \quad U_\varphi^p = U_\varphi U^p(L_x^2)$$

with norms $\|v\|_{V_\varphi^p} = \|U_\varphi(-\cdot)v\|_{V^p}$ and $\|u\|_{U_\varphi^p} = \|U_\varphi(-\cdot)u\|_{U^p}$. As for Bourgain's spaces, an immediate consequence of this definition is the equality $\|U_\varphi u_0\|_{V_\varphi^p} = \|u_0\|_{L_x^2}$, which we shall frequently use. The duality (4) gives us the estimate

$$\left\| \int_0^t U_\varphi(t-s)F(s)ds \right\|_{V_\varphi^{p'}} = \sup_{\|w\|_{U_\varphi^p} \leq 1} \left| \int_{I \times \mathbb{R}^n} F(x, t)w(x, t)dxdt \right|$$

for the solution of the inhomogeneous linear equation, cf. Lemma 3.33 in [15]. As the $X^{s,b}$ -spaces, the spaces U_φ^p admit a transfer principle. A Strichartz type estimate

$$\|U_\varphi u_0\|_{L_t^p L_x^q} \lesssim \|u_0\|_{L_x^2} \quad \text{implies} \quad \|u\|_{L_t^p L_x^q} \lesssim \|u\|_{U_\varphi^p},$$

if the order of integration is reversed, we have

$$\|U_\varphi u_0\|_{L_x^p L_t^q} \lesssim \|u_0\|_{L_x^2} \quad \text{implies} \quad \|u\|_{L_x^p L_t^q} \lesssim \|u\|_{U_\varphi^p},$$

where $r = \min(p, q)$. The U_φ^p - and U_φ^r -norms on the right can be further estimated in V_φ^2 , provided $p > 2$ and $r > 2$, due to the continuous embedding $V_{rc}^2 \subset U^{2+}$. A multilinear version of the transfer principle holds true as well, see [12, Proposition 2.19], but we will not make use of it here.

We now take $I = \mathbb{R}$ and specify φ to the phase function

$$\phi : \mathbb{R} \times \mathbb{R}^{n-1} \rightarrow \mathbb{R}, \quad (\xi, \eta) \mapsto \phi(\xi, \eta) = \xi(\xi^2 + |\eta|^2)$$

corresponding to the linear part of ZK. For $1 \leq q < \infty$ we introduce

$$\|u\|_{\dot{X}_q^s} := \left(\sum_{N \in 2^{\mathbb{Z}}} N^{sq} \|P_N u\|_{V_\phi^2}^q \right)^{\frac{1}{q}},$$

which we modify for $q = \infty$ in the usual way, i. e.

$$\|u\|_{\dot{X}_\infty^s} := \sup_{N \in 2^{\mathbb{Z}}} N^s \|P_N u\|_{V_\phi^2}.$$

Then we define the B -spaces

$$\dot{X}_q^s := \{u \in C(\mathbb{R}, \dot{B}_{2,q}^s) : \|u\|_{\dot{X}_q^s} < \infty\},$$

if $1 \leq q < \infty$, and

$$\dot{X}_\infty^s := \{u \in C(\mathbb{R}, \dot{B}_{2,\infty}^s) : \|u\|_{\dot{X}_\infty^s} < \infty, \lim_{N \rightarrow \infty} N^s \|P_N u\|_{V_\phi^2} = \lim_{N \rightarrow 0} N^s \|P_N u\|_{V_\phi^2} = 0\}.$$

By the limit conditions the latter is adapted to our data space $\dot{B}_{2,\infty}^{s,o}$. We emphasize that here and below the Littlewood-Paley projections are always applied with respect to *all* space variables, which we can fix in the form

$$P_N = \mathcal{F}_{xy}^{-1} \chi_{\{|\langle \xi, \eta \rangle| \sim N\}} \mathcal{F}_{xy}.$$

If in the above the real axis is replaced by a time interval $I = [0, T)$, we write $\dot{X}_{q,T}^s$ (instead of \dot{X}_q^s), which are for $s = s_c$ our solution spaces. Here $T = \infty$ is admitted for the global result. In the proof, and already to make our local statement precise, we need an auxiliary norm, which depends on k and on the space dimension. It is motivated by the linear estimates we shall use. Let I_x and I_y denote the Riesz potential operators of order -1 with respect to $x \in \mathbb{R}$ and $y \in \mathbb{R}^{n-1}$, respectively.

- For $n = 2$ we define $K(I_x, I_y)^\sigma = \mathcal{F}_{xy}^{-1} |3\xi^2 - \eta^2|^\sigma \mathcal{F}_{xy}$ and set

$$|P_N u|_{(k)} := N^{s_c} \|K(I_x, I_y)^{\frac{1}{8}} P_N u\|_{L_{xyt}^4} + N^{s_c} \|P_N u\|_{L_{xy}^4 L_t^6} + N^{\frac{1}{2} - \frac{5}{4k}} \|P_N u\|_{L_{xy}^4 L_t^{4k}},$$

where $s_c = 1 - \frac{2}{k}$.

- For $n = 3$ we have $s_c = \frac{3}{2} - \frac{2}{k}$ and define

$$|P_N u|_{(k)} := N^{s_c} \|I_x^{\frac{1}{10}} P_N u\|_{L_{xyt}^{\frac{15}{4}}} + N^{s_c} \|P_N u\|_{L_{xyt}^4} + N^{\frac{3}{4} - \frac{3}{2k}} \|P_N u\|_{L_{xy}^4 L_t^{6k}}.$$

From these we build up the Besov type norms $\|u\|_{(k,q)} = (\sum_{N \in 2^{\mathbb{Z}}} |P_N u|_{(k)}^q)^{\frac{1}{q}}$, with the usual modification for $q = \infty$. If the time interval is $[0, T)$ instead of \mathbb{R} , we write $|P_N u|_{(k,T)}$ and $\|u\|_{(k,q,T)}$, respectively. The linear estimates in Sections 4.2 and 5.1 imply via the transfer principle that

$$|P_N u|_{(k)} \lesssim N^{s_c} \|P_N u\|_{V_\phi^2} \quad \text{and hence} \quad \|u\|_{(k,q)} \lesssim \|u\|_{\dot{X}_q^{s_c}}.$$

Since all Hölder exponents in $|\cdot|_{(k)}$ are finite, this has the consequence, that for all $u \in \dot{X}_q^{s_c}$ we have $\lim_{T \rightarrow 0} \|u\|_{(k,q,T)} = 0$. For $q = \infty$ this is due to our assumption $\lim_{N \rightarrow \infty} N^s \|P_N u\|_{V_\phi^2} = \lim_{N \rightarrow 0} N^s \|P_N u\|_{V_\phi^2} = 0$ in the definition of $\dot{X}_\infty^{s_c}$. Now our main result takes the following shape.

Theorem 1. *Let $n \in \{2, 3\}$, $k \geq 3$ an integer and $s_c = \frac{n}{2} - \frac{2}{k}$. Assume $u_0 \in \dot{B}_{2,q}^{s_c}$, if $q < \infty$, or $u_0 \in \dot{B}_{2,\infty}^{s_c,o}$. Then*

- (1) *there exists a $T > 0$ and a unique solution $u \in \dot{X}_{q,T}^{s_c}$ of (2) with $u(0) = u_0$. Moreover, there exists a constant $C = C(k, q) > 0$, so that the lifespan of solutions can be chosen uniformly equal to T on the subset*

$$D_T := \{u_0 : \|U_\phi u_0\|_{(k,q,T)} \leq (4C)^{-k}\}$$

of the data space, and the map

$$S_T : D_T \rightarrow \dot{X}_{q,T}^{s_c}, \quad u_0 \mapsto S_T u_0 := u$$

(data upon solution) is Lipschitz continuous.

- (2) *there exists $\varepsilon = \varepsilon(k, q) > 0$ such that, if $\|u_0\|_{\dot{B}_{2,q}^{s_c}} \leq \varepsilon$, there exists a unique global solution $u \in \dot{X}_{q,\infty}^{s_c}$ of (2) with $u(0) = u_0$. The solution map S_∞ is Lipschitz continuous from the ball $B_\varepsilon := \{u_0 : \|u_0\|_{\dot{B}_{2,q}^{s_c}} \leq \varepsilon\}$ (contained in the data space) into $\dot{X}_{q,\infty}^{s_c}$.*

3. SYMMETRIZATION AND LINEAR ESTIMATES IN TWO SPACE DIMENSIONS

In [11, Section 2.1] we observed, that *in two space dimensions* the Zakharov-Kuznetsov equation can be symmetrized by a linear change of the space variables. For that purpose we fixed $\mu := 4^{-\frac{1}{3}}$, $\lambda := \sqrt{3}\mu$ and introduced

$$R_0 : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad (x, y) \mapsto (x', y') := (\mu x + \lambda y, \mu x - \lambda y)$$

as well as $Rv := v \circ R_0$. Let $u = Rv$. Then u is a solution of the Zakharov-Kuznetsov equation

$$(5) \quad \partial_t u + \partial_x (\partial_x^2 + \partial_y^2) u = c_0 \partial_x u^{k+1}$$

with initial condition $u(0) = u_0$, iff v solves

$$(6) \quad \partial_t v + (\partial_x^3 + \partial_y^3)v = \mu c_0(\partial_x + \partial_y)v^{k+1} \quad \text{and} \quad v(0) = R^{-1}u_0.$$

For more details, see [11]¹. The exponent 2 in that paper can be replaced without any other changes by $k+1$. For the study of linear estimates we may choose $c_0 = 0$ in (5) and (6). The map R introduced above defines an isomorphism on any of the spaces $H^s(\mathbb{R}^2)$, $\dot{H}^s(\mathbb{R}^2)$, $B_{p,q}^s(\mathbb{R}^2)$, $\dot{B}_{p,q}^s(\mathbb{R}^2)$ and on all mixed Lebesgue-spaces of the types $L_{xy}^p L_t^r$ and $L_t^p L_{xy}^r$. Thus especially the well-posedness theory remains unchanged if we pass over from (5) to (6).

3.1. Estimates for the linear part of the symmetrized equation, the maximal function estimate.

For the solution of (6) with $c_0 = 0$ and initial datum v_0 we write $U_{\varphi_{sym}}(t)v_0$. With familiar notation we have $U_{\varphi_{sym}}(t) = e^{-t(\partial_x^3 + \partial_y^3)}$ or, by using the Fourier transform in the space variables,

$$U_{\varphi_{sym}}(t)v_0(x, y) = \int_{\mathbb{R}^2} e^{i(x\xi + y\eta + t\varphi_{sym}(\xi, \eta))} \widehat{v_0}(\xi, \eta) d\xi d\eta,$$

where $\varphi_{sym}(\xi, \eta) = \xi^3 + \eta^3$ is the phase function associated with the symmetrized linear problem (i. e. (6) with $c_0 = 0$ and initial datum v_0). The main advantage of the symmetrization is, that it allows us to obtain the following maximal function estimate, which is global in time and avoids any technical loss of derivatives.

Proposition 1. *Let $v_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^2)$ and $f \in L_{xy}^{\frac{4}{3}} L_t^1(\mathbb{R}^3)$. Then the estimates*

$$(7) \quad \|(I_x I_y)^{-\frac{1}{2}} \int_0^t U_{\varphi_{sym}}(t-s)f(\cdot, \cdot, s) ds\|_{L_{xy}^4 L_t^\infty} \lesssim \|f\|_{L_{xy}^{\frac{4}{3}} L_t^1}$$

and

$$(8) \quad \|U_{\varphi_{sym}} v_0\|_{L_{xy}^4 L_t^\infty} \lesssim \|(I_x I_y)^{\frac{1}{4}} v_0\|_{L_{xy}^2}$$

hold true.

Proof. We use and follow the arguments in [14, Section 3]. From Lemma 3.6 in that reference we know the estimate

$$\left| \int_{-\infty}^{\infty} e^{i(x\xi + t\xi^3)} \frac{d\xi}{|\xi|^{\frac{1}{2}}} \right| \lesssim |x|^{-\frac{1}{2}},$$

where the oscillatory integral should be understood as

$$\lim_{\varepsilon \rightarrow 0, \varepsilon > 0} \int_{-\infty}^{\infty} e^{i(x\xi + t\xi^3) - \varepsilon \xi^2} \frac{d\xi}{|\xi|^{\frac{1}{2}}}.$$

This interpretation allows us to use Fubini's theorem for a double integral of this kind, and we obtain

$$\left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(x\xi + y\eta + t(\xi^3 + \eta^3))} \frac{d\xi d\eta}{|\xi \eta|^{\frac{1}{2}}} \right| \lesssim |xy|^{-\frac{1}{2}}.$$

Thus we have

$$\left| \int_{-\infty}^{\infty} (I_x I_y)^{-\frac{1}{2}} U_{\varphi_{sym}}(t-s)f(x, y, s) ds \right| \lesssim |xy|^{-\frac{1}{2}} * \int_{-\infty}^{\infty} |f(x, y, s)| ds.$$

¹We also refer to the systematic treatise on linear transformations in connection with dispersive estimates for third order equations in two space dimensions by Ben-Artzi, Koch, and Saut, in [1].

Here the convolution is done with respect to the space variables x and y , and the bound on the right is independent of t . Now the Hardy-Littlewood-Sobolev inequality is applied to obtain

$$(9) \quad \left\| \sup_{t \in \mathbb{R}} \left| \int_{-\infty}^{\infty} (I_x I_y)^{-\frac{1}{2}} U_{\varphi_{sym}}(t-s) f(\cdot, \cdot, s) ds \right| \right\|_{L^4_{xy}} \lesssim \|f\|_{L^{\frac{4}{3}}_{xy} L^1_t}.$$

Replacing f by $\chi_{[0,t]} f$ in (9), we obtain (7), finally the TT^* -argument leads to (8). \square

Next we recall the Strichartz estimates with derivative gain. From [13, Theorem 3.1] we obtain as a special case

$$(10) \quad \|(I_x I_y)^{\frac{1}{2p}} U_{\varphi_{sym}} v_0\|_{L^p_t L^q_{xy}} \lesssim \|v_0\|_{L^2_{xy}},$$

provided $2 < p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. As usual, the endpoint case $p = 2$ is excluded here. Interpolation of (8) and the $p = q = 4$ - case of (10) leads to

Corollary 1. *Let $4 \leq r \leq \infty$ and $\sigma = \frac{1}{4} - \frac{3}{2r}$. Then the estimate*

$$(11) \quad \|U_{\varphi_{sym}} v_0\|_{L^4_{xy} L^r_t} \lesssim \|(I_x I_y)^\sigma v_0\|_{L^2_{xy}}$$

holds true. Moreover we have for $6 \leq r \leq \infty$ and $s = \frac{1}{2} - \frac{3}{r}$

$$(12) \quad \|U_{\varphi_{sym}} v_0\|_{L^4_{xy} L^r_t} \lesssim \|v_0\|_{\dot{H}^s_{xy}}.$$

In order to control the derivative in the nonlinearity we will use the sharp version of the local smoothing effect, which – by the product structure of $U_{\varphi_{sym}}(t) = e^{-t\partial_x^3} e^{-t\partial_y^3}$ – can be easily deduced from the one of the Airy equation. For that purpose we fix both space variables x and y and recall from the proof of [14, Theorem 3.5] the identity

$$\|I_x e^{-t\partial_x^3} v_0(x, y)\|_{L^2_t}^2 = c^2 \|v_0(\cdot, y)\|_{L^2_x}^2.$$

Integration with respect to y gives

$$\|I_x e^{-t\partial_x^3} v_0(x, \cdot)\|_{L^2_{yt}}^2 = c \|v_0\|_{L^2_{xy}}^2,$$

which combined with the unitarity of $e^{-t\partial_y^3}$ on L^2_y leads to

$$\|I_x e^{-t(\partial_x^3 + \partial_y^3)} v_0(x, \cdot)\|_{L^2_{yt}}^2 = \|I_x e^{-t\partial_x^3} v_0(x, \cdot)\|_{L^2_{yt}}^2 = c \|v_0\|_{L^2_{xy}}^2.$$

Thus and by symmetry we have shown the identities

$$(13) \quad \|I_x U_{\varphi_{sym}} v_0\|_{L^\infty_x L^2_{yt}} = c \|v_0\|_{L^2_{xy}}$$

and

$$(14) \quad \|I_y U_{\varphi_{sym}} v_0\|_{L^\infty_y L^2_{xt}} = c \|v_0\|_{L^2_{xy}}.$$

This smoothing effect is relatively weak in so far, as we get control over I_x or I_y , respectively, but not over the full gradient in both space variables, compare with (15) below. This might be seen as the price to pay for the symmetrization.

3.2. Linear estimates for the original ZK equation in 2 D .

In the subsequent analysis we will apply the linear estimates of the previous section to treat the symmetrized gZK equation (6). Nonetheless it may be of interest to compare them with known linear estimates for the original ZK equation, and especially to trace back the Strichartz- and maximal function estimates by the aid of the transformation R from above. So let $\phi(\xi, \eta) = \xi(\xi^2 + \eta^2)$ be the phase function and $U_\phi(t)$ the propagator associated with (the linear part of) equation (5).

The local smoothing estimate

$$(15) \quad \|IU_\phi u_0\|_{L_x^\infty L_{yt}^2} \lesssim \|u_0\|_{L_{xy}^2}$$

was shown in this case by Faminskii, see [5, Theorem 2.2]. Here I denotes the Riesz potential operator of order -1 with respect to x and y , so there is the in comparison with (13) stronger gain of the full gradient in this estimate. (Since (13) is sharp and in view of the transformation R , this difference seems somewhat surprising – at least it was for the author. But it merely reflects the fact, that R is not well-behaved as a mapping on mixed Lebesgue spaces of type $L_x^p L_y^q$, if $p \neq q$.)

To convert the Strichartz- and maximal function estimates for v into estimates for $u = Rv = v \circ R_0$ in terms of $u_0 = v_0 \circ R_0$, we apply the Fourier transform to the last identity and obtain $\widehat{u}_0 = |\det R_0|^{-1} \widehat{v}_0 \circ (R_0^\top)^{-1}$, hence

$$\widehat{v}_0(\xi', \eta') = |\det R_0| \widehat{u}_0 \circ R_0^\top(\xi', \eta').$$

We set $(\xi, \eta) = R_0^\top(\xi', \eta')$ and multiply both sides by $|\xi' \eta'|^\sigma = c_\sigma |3\xi^2 - \eta^2|^\sigma$. Then

$$|\xi' \eta'|^\sigma \widehat{v}_0(\xi', \eta') = c |3\xi^2 - \eta^2|^\sigma \widehat{u}_0(\xi, \eta)$$

which can be squared and integrated with respect to $d\xi' d\eta' = |\det R_0|^{-1} d\xi d\eta$. With the Fourier multiplier

$$K(I_x, I_y)^\sigma := \mathcal{F}_{xy}^{-1} |3\xi^2 - \eta^2|^\sigma \mathcal{F}_{xy}$$

we then have $\|(I_x I_y)^\sigma v_0\|_{L_{xy}^2} = c \|K(I_x, I_y)^\sigma u_0\|_{L_{xy}^2}$. This gives the following Strichartz type inequality for u .

$$(16) \quad \|K(I_x, I_y)^{\frac{1}{2p}} U_\phi u_0\|_{L_t^p L_{xy}^q} \lesssim \|u_0\|_{L_{xy}^2},$$

provided $2 < p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. If $p = q = 4$, we can recognize this as the special case of (the dual estimate to) Theorem 1.1 in [4] by Carbery, Kenig, and Ziesler, which has been applied by Molinet and Pilod in their work [26] on the ZK equation, cf. Proposition 3.5 in that paper. In fact, our simple considerations here give a wider range of validity with a stronger gain of derivatives, if $p \rightarrow 2$. Again the endpoint $p = 2$ is excluded.

Similarly, we have the maximal function estimate

$$(17) \quad \|U_\phi u_0\|_{L_{xy}^4 L_t^\infty} \lesssim \|K(I_x, I_y)^{\frac{1}{4}} u_0\|_{L_{xy}^2} \leq \|u_0\|_{\dot{H}_{xy}^{\frac{1}{2}}}.$$

A Sobolev embedding in the y -Variable gives

$$(18) \quad \|U_\phi u_0\|_{L_x^4 L_{yt}^\infty} \lesssim \|K(I_x, I_y)^{\frac{1}{4}} J_y^{\frac{1}{4}+} u_0\|_{L_{xy}^2},$$

which is comparable with Proposition 1.5 in [22] by Linares and Pastor. see also Corollary 2.7 of [23]. The advantage here is, that (18) holds globally in time.

4. DISCUSSION OF THE 2 D - CASE

Throughout this section we have $n = 2$ and hence $s_c = 1 - \frac{2}{k}$. $U_{\varphi_{sym}}^p$ and $V_{\varphi_{sym}}^p$ will denote the U^p - and V^p - spaces associated with the phase function

$$\varphi_{sym} : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad (\xi, \eta) \mapsto \varphi_{sym}(\xi, \eta) = \xi^3 + \eta^3.$$

The solution space $\dot{X}_q^{s_c}$ is that one with norm built on $V_{\varphi_{sym}}^2$.

4.1. The central multilinear estimate.

We will start with a multilinear estimate on dyadic pieces of functions $v_1, \dots, v_{k+1} \in V_{\varphi_{sym}}^2$. We recall the quantity

$$|P_N v|_{(k)} := N^{s_c} \|(I_x I_y)^{\frac{1}{8}} P_N v\|_{L_{xyt}^4} + N^{s_c} \|P_N v\|_{L_{xy}^4 L_t^6} + N^{\frac{1}{2} - \frac{5}{4k}} \|P_N v\|_{L_{xy}^4 L_t^{4k}}.$$

We remark that by the transfer principle and the linear estimates (11) and (12) all three contributions are bounded by the $V_{\varphi_{sym}}^2$ - norm, more precisely we have

$$|P_N v|_{(k)} \lesssim N^{s_c} \|P_N v\|_{V_{\varphi_{sym}}^2}.$$

If the mixed norms are replaced by $L_{xy}^4 L_t^r$, which means that the integration is restricted to the time interval $(0, T)$, we write $|P_N v|_{(k, T)}$ for the corresponding composed quantity. Here we may have $T = \infty$. Observe that $\lim_{T \rightarrow 0} |P_N v|_{(k, T)} = 0$, whenever $v \in V_{\varphi_{sym}}^2$.

Lemma 1. *Let $v_1, \dots, v_{k+1} \in V_{\varphi_{sym}}^2$, $w \in U_{\varphi_{sym}}^2$ with $\|w\|_{U_{\varphi_{sym}}^2} \leq 1$, N, N_1, \dots, N_{k+1} dyadic numbers with $N_1 \leq N_2 \leq \dots \leq N_{k+1}$ and $N \lesssim N_{k+1}$. Then there exists $\varepsilon > 0$ such that*

$$N^{s_c} \left| \int_{\mathbb{R}^3} P_{N_1} v_1 \cdot \dots \cdot P_{N_{k+1}} v_{k+1} \cdot \partial_x P_N w dx dy dt \right| \lesssim N^\varepsilon N_1^\varepsilon N_{k+1}^{-2\varepsilon} \prod_{j=1}^{k+1} |P_{N_j} v_j|_{(k)}.$$

The same holds true, if ∂_x is replaced by ∂_y .

Proof. We consider three cases, depending on the relative sizes of the spatial frequencies (ξ_{k+1}, η_{k+1}) and (ξ, η) of v_{k+1} and w , respectively. Observe that by our assumptions $|(\xi, \eta)| \lesssim |(\xi_{k+1}, \eta_{k+1})|$. In the sequel, let ε be a positive number, which has to be chosen sufficiently small in dependence of k .

Case 1: $|\eta_{k+1}| \lesssim |\xi_{k+1}|$. Here we may replace the factor $P_{N_{k+1}} v_{k+1}$ by $N_{k+1}^{-\varepsilon} I_x^\varepsilon P_{N_{k+1}} v_{k+1}$.

Case 2: $|\xi_{k+1}| \ll |\eta_{k+1}|$ and $|\xi| \lesssim |\eta|$. Here we may replace $(P_{N_{k+1}} v_{k+1})(\partial_x P_N w)$ by $(N_{k+1}^{-\varepsilon} I_y^\varepsilon P_{N_{k+1}} v_{k+1})(\partial_y P_N w)$ and argue as in Case 1 with the roles of x and y (respectively of the ξ 's and η 's) interchanged.

Case 3: $|\xi_{k+1}| \ll |\eta_{k+1}|$ and $|\eta| \lesssim |\xi|$. Since in this case $|\xi| \lesssim N_{k+1} \sim |\eta_{k+1}|$, we have $|\eta| \ll |\eta_{k+1}|$. By the convolution constraint $|\sum_{j=1}^{k+1} \eta_j| = |\eta|$, there exists at least one $j \in \{1, \dots, k\}$ with $|\eta_j| \sim |\eta_{k+1}|$. This implies $N_K \sim N_{k+1}$ and especially $N \lesssim N_k$.

Treatment of Case 1: The contribution from this case is bounded by

$$\begin{aligned} & N^\varepsilon N_{k+1}^{s_c - 2\varepsilon} \left| \int_{\mathbb{R}^3} P_{N_1} v_1 \cdot \dots \cdot P_{N_k} v_k \cdot (I_x^\varepsilon P_{N_{k+1}} v_{k+1}) \cdot \partial_x P_N w dx dy dt \right| \\ & \lesssim N^\varepsilon N_{k+1}^{s_c - 2\varepsilon} \|P_{N_1} v_1 \cdot \dots \cdot P_{N_k} v_k \cdot (I_x^\varepsilon P_{N_{k+1}} v_{k+1})\|_{L_x^1 L_{yt}^2} \|\partial_x P_N w\|_{L_x^\infty L_{yt}^2}, \end{aligned}$$

where by the local smoothing effect (13) and the transfer principle the last factor is bounded by $\|P_N w\|_{U_{\varphi_{sym}}^2} \leq 1$. For $\varepsilon > 0$ sufficiently small we choose

$$\frac{1}{p_1} = \frac{3}{4k}, \quad \frac{1}{q_1} = \frac{1}{4k} + \varepsilon, \quad \frac{1}{r_1} = \frac{1}{3k} - \frac{2\varepsilon}{3},$$

and, for $j \in \{2, \dots, k\}$,

$$\frac{1}{p_j} = \frac{3}{4k}, \quad \frac{1}{q_j} = \frac{1}{4k}, \quad \frac{1}{r_j} = \frac{1}{3k},$$

as well as

$$\frac{1}{p_{k+1}} = \frac{1}{4}, \quad \frac{1}{q_{k+1}} = \frac{1}{4} - \varepsilon, \quad \frac{1}{r_{k+1}} = \frac{1}{6} + \frac{2\varepsilon}{3}.$$

Since $k \geq 3$, we have $p_j \geq 4$ and $q_j \geq 4$ for all $j \in \{1, \dots, k+1\}$. Moreover $\sum_{j=1}^{k+1} \frac{1}{p_j} = 1$ and $\sum_{j=1}^{k+1} \frac{1}{q_j} = \sum_{j=1}^{k+1} \frac{1}{r_j} = \frac{1}{2}$, so that Hölder's inequality gives the upper bound

$$(19) \quad N^\varepsilon N_{k+1}^{-2\varepsilon} \left(\prod_{j=1}^k \|P_{N_j} v_j\|_{L_x^{p_j} L_y^{q_j} L_t^{r_j}} \right) N_{k+1}^{s_c} \|I_x^\varepsilon P_{N_{k+1}} v_{k+1}\|_{L_x^{p_{k+1}} L_y^{q_{k+1}} L_t^{r_{k+1}}}.$$

For the v_1 - factor we use Sobolev embeddings in the space variables to obtain

$$\|P_{N_1} v_1\|_{L_x^{p_1} L_y^{q_1} L_t^{r_1}} \lesssim N_1^{\frac{1}{2} - \frac{1}{k} - \varepsilon} \|P_{N_1} v_1\|_{L_{xy}^4 L_t^{r_1}} = N_1^\varepsilon N_1^{\frac{1}{2} - \frac{1}{k} - 2\varepsilon} \|P_{N_1} v_1\|_{L_{xy}^4 L_t^{r_1}}.$$

We choose θ so that $\frac{1}{r_1} = \frac{1-\theta}{6} + \frac{\theta}{4k}$. Then Lyapunov's inequality gives the bound

$$N_1^{\frac{1}{2} - \frac{1}{k} - 2\varepsilon} \|P_{N_1} v_1\|_{L_{xy}^4 L_t^{r_1}} \leq \left(N_1^{s_c} \|P_{N_1} v_1\|_{L_{xy}^4 L_t^6} \right)^{1-\theta} \left(N_1^{\frac{1}{2} - \frac{5}{4k}} \|P_{N_1} v_1\|_{L_{xy}^4 L_t^{4k}} \right)^\theta,$$

which in turn is dominated by

$$N_1^{s_c} \|P_{N_1} v_1\|_{L_{xy}^4 L_t^6} + N_1^{\frac{1}{2} - \frac{5}{4k}} \|P_{N_1} v_1\|_{L_{xy}^4 L_t^{4k}} \leq |P_{N_1} v_1|_{(k)}.$$

Collecting terms we obtain

$$\|P_{N_1} v_1\|_{L_x^{p_1} L_y^{q_1} L_t^{r_1}} \lesssim N_1^\varepsilon |P_{N_1} v_1|_{(k)},$$

and, taking $\varepsilon = 0$ in this calculation for v_1 , we as well have for $j \in \{2, \dots, k\}$ that

$$\|P_{N_j} v_j\|_{L_x^{p_j} L_y^{q_j} L_t^{r_j}} \lesssim |P_{N_j} v_j|_{(k)}.$$

For the last factor we use a Sobolev embedding with respect to the y - variable and a convexity inequality (like Lyapunov's inequality above) to obtain

$$\begin{aligned} \|I_x^\varepsilon P_{N_{k+1}} v_{k+1}\|_{L_x^{p_{k+1}} L_y^{q_{k+1}} L_t^{r_{k+1}}} &\lesssim \|(I_x I_y)^\varepsilon P_{N_{k+1}} v_{k+1}\|_{L_{xy}^4 L_t^{r_{k+1}}} \\ &\lesssim \|(I_x I_y)^{\frac{1}{8}} P_{N_{k+1}} v_{k+1}\|_{L_{xy}^4} + \|P_{N_{k+1}} v_{k+1}\|_{L_{xy}^4 L_t^6}, \end{aligned}$$

so that

$$N_{k+1}^{s_c} \|I_x^\varepsilon P_{N_{k+1}} v_{k+1}\|_{L_x^{p_{k+1}} L_y^{q_{k+1}} L_t^{r_{k+1}}} \lesssim |P_{N_{k+1}} v_{k+1}|_{(k)}.$$

Summarizing the estimates for the single factors, we see that (19) is in fact bounded by

$$N^\varepsilon N_1^\varepsilon N_{k+1}^{-2\varepsilon} \prod_{j=1}^{k+1} |P_{N_j} v_j|_{(k)},$$

as desired.

Estimation for Case 3: Since $N_k \sim N_{k+1}$ here, the contribution is bounded by

$$N^\varepsilon N_k^\varepsilon N_{k+1}^{s_c - 2\varepsilon} \|P_{N_1} v_1 \cdot \dots \cdot P_{N_{k+1}} v_{k+1}\|_{L_x^1 L_{yt}^2}.$$

For $j \in \{1, \dots, k-1\}$ we choose Hölder exponents p_j , q_j , and r_j precisely as in Case 1. Moreover we set

$$\frac{1}{p_k} = \frac{3}{4k}, \quad \frac{1}{q_k} = \frac{1}{4k} - \varepsilon, \quad \frac{1}{r_k} = \frac{1}{3k} + \frac{2\varepsilon}{3},$$

as well as

$$\frac{1}{p_{k+1}} = \frac{1}{4}, \quad \frac{1}{q_{k+1}} = \frac{1}{4}, \quad \frac{1}{r_{k+1}} = \frac{1}{6}.$$

Then again we have $p_j \geq 4$ and $q_j \geq 4$ for all $j \in \{1, \dots, k+1\}$ and $\sum_{j=1}^{k+1} \frac{1}{p_j} = 1$

as well as $\sum_{j=1}^{k+1} \frac{1}{q_j} = \sum_{j=1}^{k+1} \frac{1}{r_j} = \frac{1}{2}$. Hölder's inequality gives the only slightly different upper bound

$$(20) \quad N^\varepsilon N_k^\varepsilon N_{k+1}^{-2\varepsilon} \left(\prod_{j=1}^k \|P_{N_j} v_j\|_{L_x^{p_j} L_y^{q_j} L_t^{r_j}} \right) N_{k+1}^{s_c} \|P_{N_{k+1}} v_{k+1}\|_{L_{xy}^4 L_t^6}.$$

From the estimates concerning Case 1 we already know that

$$(21) \quad \prod_{j=1}^{k-1} \|P_{N_j} v_j\|_{L_x^{p_j} L_y^{q_j} L_t^{r_j}} \lesssim N_1^\varepsilon \prod_{j=1}^{k-1} |P_{N_j} v_j|_{(k)}.$$

Moreover it is clear by our choices, that

$$N_{k+1}^{s_c} \|P_{N_{k+1}} v_{k+1}\|_{L_{xy}^4 L_t^6} \leq |P_{N_{k+1}} v_{k+1}|_{(k)},$$

and it remains to estimate the factor for $j = k$. Sobolev embeddings in x and y give

$$\|P_{N_k} v_k\|_{L_x^{p_k} L_y^{q_k} L_t^{r_k}} \lesssim N_k^{\frac{1}{2} - \frac{1}{k} + \varepsilon} \|P_{N_k} v_k\|_{L_{xy}^4 L_t^{r_k}} = N_k^{-\varepsilon} N_k^{\frac{1}{2} - \frac{1}{k} + 2\varepsilon} \|P_{N_k} v_k\|_{L_{xy}^4 L_t^{r_k}}.$$

Applying Lyapunov's inequality again we obtain (replace ε by $-\varepsilon$ in the corresponding argument for v_1 in Case 1)

$$N_k^{\frac{1}{2} - \frac{1}{k} + 2\varepsilon} \|P_{N_k} v_k\|_{L_{xy}^4 L_t^{r_k}} \lesssim N_k^{s_c} \|P_{N_k} v_k\|_{L_{xy}^4 L_t^6} + N_k^{\frac{1}{2} - \frac{5}{4k}} \|P_{N_k} v_k\|_{L_{xy}^4 L_t^{4k}} \lesssim |P_{N_k} v_k|_{(k)},$$

so that

$$(22) \quad \|P_{N_k} v_k\|_{L_x^{p_k} L_y^{q_k} L_t^{r_k}} \lesssim N_k^{-\varepsilon} |P_{N_k} v_k|_{(k)}.$$

The comparison of (21) and (22) shows, that we have successfully exchanged the large factor N_k^ε by the smaller N_1^ε . Altogether

$$(20) \lesssim N^\varepsilon N_1^\varepsilon N_{k+1}^{-2\varepsilon} \prod_{j=1}^{k+1} |P_{N_j} v_j|_{(k)}.$$

This completes the estimation in Case 3.

The statement about ∂_y instead of ∂_x is obvious by symmetry. \square

The next step is to sum up these estimates on dyadic pieces, which will necessarily involve the auxiliary norms

$$\|v\|_{(k,q)} = \left(\sum_{N \in 2^{\mathbb{Z}}} |P_N v|_{(k)} \right)^{\frac{1}{q}}$$

with the usual modification for $q = \infty$. (We write $\|v\|_{(k,q,T)}$, if these norms are assembled from $(|P_N v|_{(k,T)})_{N \in 2^{\mathbb{Z}}}$.) They remain finite for $v \in \dot{X}_q^{sc}$ (or $v \in \dot{X}_{q,T}^{sc}$, respectively). For $v_1, \dots, v_{k+1} \in \dot{X}_q^{sc}$ we introduce

$$F(v_1, \dots, v_{k+1})(t) := \int_0^t U_{\varphi_{sym}}(t-s)(\partial_x)(v_1 \cdot \dots \cdot v_{k+1})(s) ds,$$

the dependence on x and y was suppressed here.

Lemma 2. *For $v_1, \dots, v_{k+1} \in \dot{X}_q^{sc}$ we have $F(v_1, \dots, v_{k+1}) \in \dot{X}_q^{sc}$ and the estimate*

$$\|F(v_1, \dots, v_{k+1})\|_{\dot{X}_q^{sc}} \lesssim \prod_{j=1}^{k+1} \|v_j\|_{(k,q)}$$

holds true.

Remark: The statement is still correct, if \dot{X}_q^{sc} and $\|\cdot\|_{(k,q)}$ are replaced by $\dot{X}_{q,T}^{sc}$ and $\|\cdot\|_{(k,q,T)}$, respectively, and if ∂_x is changed into ∂_y .

Proof. By the duality between V_{φ}^2 and U_{φ}^2 (cf. (4) and the subsequent remark) we have

$$\|F(v_1, \dots, v_{k+1})\|_{V_{\varphi_{sym}}^2} = \sup_{\|w\|_{U_{\varphi_{sym}}^2} \leq 1} \left| \int_{\mathbb{R}^3} v_1 \cdot \dots \cdot v_{k+1} \partial_x w dx dy dt \right|.$$

Thus Lemma 1 tells us that for $N_1 \leq \dots \leq N_{k+1}$

$$N^{sc} \|P_N F(P_{N_1} v_1, \dots, P_{N_{k+1}} v_{k+1})\|_{V_{\varphi_{sym}}^2} \lesssim N^{\varepsilon} N_1^{\varepsilon} N_{k+1}^{-2\varepsilon} \prod_{j=1}^{k+1} |P_{N_j} v_j|_{(k)}.$$

(Here, by the convolution constraint $|(\xi, \eta)| = |\sum_{j=1}^{k+1} (\xi_j, \eta_j)|$ we have only contributions for $N \lesssim N_{k+1}$.) We fix N and $N_{k+1} \gtrsim N$ and sum up the geometric series in $N_1 \leq \dots \leq N_k$. This gives

$$\begin{aligned} & \sum_{\substack{N_1 \leq \dots \leq N_k \\ N_k \leq N_{k+1}}} N^{sc} \|P_N F(P_{N_1} v_1, \dots, P_{N_{k+1}} v_{k+1})\|_{V_{\varphi_{sym}}^2} \\ (23) \quad & \lesssim N^{\varepsilon} N_{k+1}^{-\varepsilon} \left(\prod_{j=1}^k \|v_j\|_{(k,\infty)} \right) |P_{N_{k+1}} v_{k+1}|_{(k)}. \end{aligned}$$

Now we distinguish between $q = \infty$ and $q < \infty$.

Case 1: $q = \infty$. Here we simply sum up one last geometric series in $N_{k+1} \gtrsim N$, which leads to

$$(24) \quad \sum_{\substack{N_1 \leq \dots \leq N_{k+1} \\ N \lesssim N_{k+1}}} N^{sc} \|P_N F(P_{N_1} v_1, \dots, P_{N_{k+1}} v_{k+1})\|_{V_{\varphi_{sym}}^2} \lesssim \prod_{j=1}^{k+1} \|v_j\|_{(k,\infty)}.$$

Since this works for all orders of N_1, \dots, N_{k+1} , we have for N fixed by the triangle inequality

$$N^{sc} \|P_N F(v_1, \dots, v_{k+1})\|_{V_{\varphi_{sym}}^2} \lesssim \prod_{j=1}^{k+1} \|v_j\|_{(k,\infty)}.$$

Taking the supremum over all $N \in 2^{\mathbb{Z}}$ we have achieved the claimed inequality in the case $q = \infty$.

Going back to (24) we see that, since the sums over N_1, \dots, N_{k+1} exist,

$$\lim_{N \rightarrow \infty} \sum_{\substack{N_1 \leq \dots \leq N_{k+1} \\ N \lesssim N_{k+1}}} N^{s_c} \|P_N F(P_{N_1} v_1, \dots, P_{N_{k+1}} v_{k+1})\|_{V_{\varphi_{sym}}^2} = 0,$$

which implies $\lim_{N \rightarrow \infty} N^{s_c} \|P_N F(v_1, \dots, v_{k+1})\|_{V_{\varphi_{sym}}^2} = 0$. To see that the limit for $N \rightarrow 0$ vanishes, too, let $\delta > 0$ be given. Then there exists $N_\delta \in 2^{\mathbb{Z}}$ such that $|P_{N_{k+1}} v_{k+1}|_{(k)} \leq \delta$ for all $N_{k+1} \leq N_\delta$. Thus (cf. the right hand side of (23))

$$\sum_{N_{k+1} \gtrsim N} \frac{N^\varepsilon}{N_{k+1}^\varepsilon} \left(\prod_{j=1}^k \|v_j\|_{(k, \infty)} \right) |P_{N_{k+1}} v_{k+1}|_{(k)} \lesssim \delta \prod_{j=1}^k \|v_j\|_{(k, \infty)} + \frac{N^\varepsilon}{N_\delta^\varepsilon} \prod_{j=1}^{k+1} \|v_j\|_{(k, \infty)}.$$

Now $N \rightarrow 0$, then $\delta \rightarrow 0$.

To close the discussion in the case $q = \infty$, we have to show that $F(v_1, \dots, v_{k+1}) \in C(\mathbb{R}, \dot{B}_{2, \infty}^{s_c, o})$. For that purpose we fix $t_0 \in \mathbb{R}$ and denote the characteristic function of the t -intervall between t_0 and $t_0 + h$ by χ_h (h may be negative). Then, by the continuous embedding $\dot{X}_\infty^{s_c} \subset C(\mathbb{R}, \dot{B}_{2, \infty}^{s_c, o})$ and the estimate already shown we have

$$\begin{aligned} & \|F(v_1, \dots, v_{k+1})(t_0 + h) - F(v_1, \dots, v_{k+1})(t_0)\|_{\dot{B}_{2, \infty}^{s_c}} \\ & \leq \sup_{t \in \mathbb{R}} \|F(\chi_h v_1, \dots, \chi_h v_{k+1})(t)\|_{\dot{B}_{2, \infty}^{s_c}} \\ & \leq \|F(\chi_h v_1, \dots, \chi_h v_{k+1})\|_{\dot{X}_\infty^{s_c}} \lesssim \prod_{j=1}^{k+1} \|\chi_h v_j\|_{(k, \infty)}, \end{aligned}$$

which tends to zero with $h \rightarrow 0$.

Case 2: $q < \infty$. We sum up the right hand side of (23) in N_{k+1} using Hölder's inequality. This gives

$$\begin{aligned} & \sum_{N_{k+1} \gtrsim N} \frac{N^\varepsilon}{N_{k+1}^\varepsilon} \left(\prod_{j=1}^k \|v_j\|_{(k, \infty)} \right) |P_{N_{k+1}} v_{k+1}|_{(k)} \\ & \lesssim \prod_{j=1}^k \|v_j\|_{(k, \infty)} \left(\sum_{N_{k+1} \gtrsim N} N^{\frac{q\varepsilon}{2}} N_{k+1}^{-\frac{q\varepsilon}{2}} |P_{N_{k+1}} v_{k+1}|_{(k)}^q \right)^{\frac{1}{q}}. \end{aligned}$$

Now we can take the $\ell_N^q(2^{\mathbb{Z}})$ -norm of this and sum up first in $N \lesssim N_{k+1}$ and then in N_{k+1} to obtain

$$\begin{aligned} & \left(\sum_{N \in 2^{\mathbb{Z}}} N^{s_c q} \left\| \sum_{\substack{N_1 \leq \dots \leq N_{k+1} \\ N \lesssim N_{k+1}}} P_N F(P_{N_1} v_1, \dots, P_{N_{k+1}} v_{k+1}) \right\|_{V_{\varphi_{sym}}^2}^q \right)^{\frac{1}{q}} \\ & \lesssim \left(\prod_{j=1}^k \|v_j\|_{(k, \infty)} \right) \|v_{k+1}\|_{(k, q)} \lesssim \prod_{j=1}^{k+1} \|v_j\|_{(k, q)}, \end{aligned}$$

where in the last step the continuous embedding $\ell^q \subset \ell^\infty$ was used. The same bound holds for all orders of N_1, \dots, N_{k+1} , hence we get the claimed inequality. The continuity follows by the same arguments as in Case 1. \square

4.2. Well-posedness for the symmetrized equation.

Here we prove the local and global well-posedness of the Cauchy problem for (6) with initial data v_0 in Besov spaces of critical regularity. By the discussion about symmetrization at the beginning of Section 3.1 this implies the two dimensional part of Theorem 1.

Theorem 2. *Let $v_0 \in \dot{B}_{2,q}^{s_c}$, if $q < \infty$, or $v_0 \in \dot{B}_{2,\infty}^{s_c,o}$. Then*

- (1) *there exists a $T > 0$ and a unique solution $v \in \dot{X}_{q,T}^{s_c}$ of (6) with $v(0) = v_0$. Moreover, there exists a constant $C = C(k, q) > 0$, so that the lifespan of solutions can be chosen uniformly equal to T on the subset*

$$D_T := \{v_0 : \|U_{\varphi_{sym}} v_0\|_{(k,q,T)} \leq (4C)^{-k}\}$$

of the data space, and the map

$$S_T : D_T \rightarrow \dot{X}_{q,T}^{s_c}, \quad v_0 \mapsto S_T v_0 := v$$

(data upon solution) is Lipschitz continuous.

- (2) *there exists $\varepsilon = \varepsilon(k, q) > 0$ such that, if $\|v_0\|_{\dot{B}_{2,q}^{s_c}} \leq \varepsilon$, there exists a unique global solution $v \in \dot{X}_{q,\infty}^{s_c}$ of (6) with $v(0) = v_0$. The solution map S_∞ is Lipschitz continuous from the ball $B_\varepsilon := \{v_0 : \|v_0\|_{\dot{B}_{2,q}^{s_c}} \leq \varepsilon\}$ (contained in the data space) into $\dot{X}_{q,\infty}^{s_c}$.*

Proof. For given v_0 we search a solution $v = \psi + w$, where $\psi = U_{\varphi_{sym}} v_0$ is a solution of the linear equation with $\psi(0) = v_0$ and w solves the integral equation $w = \Lambda_\psi w$ defined by

$$\Lambda_\psi w(t) = \int_0^t U_{\varphi_{sym}}(t-s)(\partial_x + \partial_y)(w + \psi)^{k+1}(s) ds.$$

For $j \in \{1, 2\}$ let $v_0^{(j)} \in \dot{B}_{2,q}^{s_c}$ (respectively $v_0^{(j)} \in \dot{B}_{2,\infty}^{s_c,o}$) and $\psi_j = U_{\varphi_{sym}} v_0^{(j)}$ with $\|\psi_j\|_{(k,q,T)} \leq R_0$ as well as $w_j \in \dot{X}_{q,T}^{s_c}$ with $\|w_j\|_{\dot{X}_{q,T}^{s_c}} \leq R$. (The relation between T , R_0 and R will be specified within the next few lines.) Then by Lemma 2 and some elementary estimates we obtain

$$(25) \quad \|\Lambda_{\psi_1} w_1 - \Lambda_{\psi_2} w_2\|_{\dot{X}_{q,T}^{s_c}} \leq C(R^k + R_0^k)(\|w_1 - w_2\|_{\dot{X}_{q,T}^{s_c}} + \|\psi_1 - \psi_2\|_{(k,q,T)})$$

with a constant C , which may only depend on k and q . Especially for $w_2 = \psi_2 = 0$ we see that

$$(26) \quad \|\Lambda_{\psi_1} w_1\|_{\dot{X}_{q,T}^{s_c}} \leq C(R^k + R_0^k)(R + R_0),$$

if we take $\psi_1 = \psi_2$ in (25), we get

$$(27) \quad \|\Lambda_{\psi_1} w_1 - \Lambda_{\psi_1} w_2\|_{\dot{X}_{q,T}^{s_c}} \leq C(R^k + R_0^k)\|w_1 - w_2\|_{\dot{X}_{q,T}^{s_c}}.$$

Now we fix $R = R_0$ in that way, that $CR^k = CR_0^k = \frac{1}{4}$. Since for any $v_0 \in \dot{B}_{2,q}^{s_c}$ (respectively $v_0 \in \dot{B}_{2,\infty}^{s_c,o}$) we have $\lim_{T \rightarrow 0} \|U_{\varphi_{sym}} v_0\|_{(k,q,T)}$, we can reach $\|\psi_j\|_{(k,q,T)} \leq R_0$ by choosing T small enough. With this choice we have

$$\|\Lambda_{\psi_1} w_1\|_{\dot{X}_{q,T}^{s_c}} \leq R \quad \text{and} \quad \|\Lambda_{\psi_1} w_1 - \Lambda_{\psi_1} w_2\|_{\dot{X}_{q,T}^{s_c}} \leq \frac{1}{2} \|w_1 - w_2\|_{\dot{X}_{q,T}^{s_c}}.$$

Moreover, we know from Lemma 2 that - for $w \in \dot{X}_{q,T}^{s_c}$ - $\Lambda_{\psi_1} w \in \dot{X}_{q,T}^{s_c}$, especially it is a continuous function with values in the data space. Thus for fixed ψ_1 the mapping Λ_{ψ_1} is a contraction of the closed ball of radius R in $\dot{X}_{q,T}^{s_c}$ into itself. The contraction mapping principle provides a solution of $\Lambda_{\psi_1} w = w$, which is unique in this ball. Since for any $w \in \dot{X}_{q,T}^{s_c}$ we have $\lim_{T \rightarrow 0} \|w\|_{(k,q,T)} = 0$, we can use a standard argument, to extend the uniqueness property to the whole $\dot{X}_{q,T}^{s_c}$. The statement about the lifespan merely reflects our choices. These also give, if inserted into (25) the inequality

$$\|\Lambda_{\psi_1} w_1 - \Lambda_{\psi_2} w_2\|_{\dot{X}_{q,T}^{s_c}} \leq \frac{1}{2} \|w_1 - w_2\|_{\dot{X}_{q,T}^{s_c}} + \frac{1}{2} \|\psi_1 - \psi_2\|_{(k,q,T)},$$

which for solutions $w_1 = \Lambda_{\psi_1} w_1$ and $w_2 = \Lambda_{\psi_2} w_2$ implies the Lipschitz bound

$$\|w_1 - w_2\|_{\dot{X}_{q,T}^{s_c}} \leq \|\psi_1 - \psi_2\|_{(k,q,T)} \lesssim \|v_0^{(1)} - v_0^{(2)}\|_{\dot{B}_{2,q}^{s_c}}.$$

Clearly, if $v_j = \psi_j + w_j$, we have the same (up to a factor) upper bound for $\|v_1 - v_2\|_{\dot{X}_{q,T}^{s_c}}$. Now the local part of the Theorem is shown. The global part is similar: One uses $\|\psi_j\|_{(k,q,\infty)} \lesssim \|v_0^{(j)}\|_{\dot{B}_{2,q}^{s_c}}$ and replaces R_0 by ε in the inequalities. We omit further details. \square

5. MODIFICATIONS IN THE 3 D - CASE

5.1. Linear estimates in 3 D .

The linear part of the ZK equation in 3 D is

$$(28) \quad u_t + \partial_x \Delta u = 0,$$

where the Laplacian can be written as $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}$ in order to emphasize the symmetry in the second and third space variable. The phase function corresponding to (28) is

$$\phi(\xi, \eta) = \xi(\xi^2 + |\eta|^2) \quad \text{with} \quad (\xi, \eta) = (\xi, \eta_1, \eta_2) \in \mathbb{R}^3.$$

Let $(U_\phi(t))_{t \in \mathbb{R}}$ denote the associated unitary group, so that solutions u of (28) with initial datum u_0 become $u(t, x, y) = U_\phi(t)u_0(x, y)$. Then we can rely on various known linear estimates for such solutions. In order to control the derivative in the multilinear estimates we may use the local smoothing effect of Kato type, i.e.

$$(29) \quad \|IU_\phi u_0\|_{L_x^\infty L_{yt}^2} \lesssim \|u_0\|_{L_{xy}^2}.$$

Here I denotes the Riesz potential operator of order -1 with respect to all space variables. The proof of (29) follows the same lines as in the 2 D case, the calculation is carried out by Ribaud and Vento in [28, Proposition 3.1]. On the other hand we have the following Strichartz type estimates due to Linares and Saut.

Lemma 3. *Let $\frac{1}{4} \leq \frac{1}{p} < \frac{2}{7}$ and $s = \frac{6}{p} - \frac{3}{2}$. Then*

$$(30) \quad \|I_x^s U_\phi u_0\|_{L_{xyt}^p} \lesssim \|u_0\|_{L_{xy}^2}.$$

The derivative gain here involves only the x - variable, not the full gradient. For $p < 4$ this estimate is the special case of [24, Proposition 3.1], where $p = q$.² The case $p = q = 4$, which will play a major role in our considerations, can be obtained by similar arguments. An alternative approach (allowing a bilinear refinement) was sketched in Section 2 of [10].

A problem seems to occur, if we try to prove an appropriate maximal function estimate (global in time and even without an ε unnecessary derivative loss), since the symmetrization argument we applied successfully in 2 D fails in three space dimensions. Nonetheless, let us for a short heuristic consider the symmetric phase function

$$\tilde{\phi}(\xi, \eta_1, \eta_2) = \xi^3 + \eta_1^3 + \eta_2^3.$$

Then the argument in the proof of Proposition 1 gives the bound

$$\|U_{\tilde{\phi}} u_0\|_{L_{xy}^4 L_t^\infty} \lesssim \|I_x^{\frac{1}{4}} I_y^{\frac{1}{2}} u_0\|_{L_{xy}^2},$$

²The regularity gain in the $p = q$ - version written down here is restricted by $s < \frac{3}{14}$, the nonsymmetric version is stronger and exhibits a gain of up to $\frac{3}{8}$ - derivatives, see [24]. For our purposes an I_x^ε will do, but this ε is essential in our treatment of the quartic nonlinearity.

which shows, what we may expect: The loss of $\frac{3}{4}$ derivatives in an $L_{xy}^4 L_t^\infty$ - estimate. It turns out that a fairly soft argument combined with the Strichartz type estimate (30) will give us an appropriate substitute. This works, since we are in three dimensions and the phase function is cubic.

Lemma 4. *Assume $0 < \frac{1}{q} \leq \frac{1}{p} < \frac{2}{7}$ and $\frac{1}{q} + \frac{5}{p} \leq \frac{3}{2}$. Then for $s = 3(\frac{1}{2} - \frac{1}{p} - \frac{1}{q})$ we have*

$$(31) \quad \|U_\phi u_0\|_{L_{xy}^p L_t^q} \lesssim \|u_0\|_{\dot{H}_{xy}^s}.$$

Proof. By Sobolev embedding in the space variables we may assume $p \leq 4$. Let $u = U_\phi u_0$. Then for the space-time Fourier transform of u we have

$$\mathcal{F}u(\xi, \eta, \tau) = \delta_0(\tau - \phi(\xi, \eta)) \mathcal{F}_{xy} u_0(\xi, \eta),$$

so that $\tau = \phi(\xi, \eta) = \xi(\xi^2 + |\eta|^2)$ in the support of $\mathcal{F}u$. Now if $p \leq q < \infty$ we can apply a Sobolev embedding in the time variable to obtain with $s_p = \frac{6}{p} - \frac{3}{2}$ as in Lemma 3

$$\begin{aligned} \|U_\phi u_0\|_{L_{xy}^p L_t^q} &\lesssim \|\mathcal{F}^{-1} |\tau|^{\frac{1}{p} - \frac{1}{q}} \mathcal{F} U_\phi u_0\|_{L_{xyt}^p} = \|\mathcal{F}^{-1} |\xi(\xi^2 + |\eta|^2)|^{\frac{1}{p} - \frac{1}{q}} \mathcal{F} U_\phi u_0\|_{L_{xyt}^p} \\ &\lesssim \|\mathcal{F}_{xy}^{-1} |\xi|^{\frac{1}{p} - \frac{1}{q} - s_p} (\xi^2 + |\eta|^2)^{\frac{1}{p} - \frac{1}{q}} \mathcal{F}_{xy} u_0\|_{L_{xy}^2}, \end{aligned}$$

where in the last step we have applied (30). The assumption $\frac{1}{q} + \frac{5}{p} \leq \frac{3}{2}$ implies that $\frac{1}{p} - \frac{1}{q} - s_p \geq 0$, so that the Fourier multiplier can be estimated by $(\xi^2 + |\eta|^2)^{\frac{s}{2}}$. \square

5.2. The multilinear estimate on dyadic pieces in 3 D .

Here we prove the estimate on dyadic pieces in three dimensions, which corresponds to Lemma 1 in Section 4.1. This will look like a copy, but there are differences. We fix $s_c = \frac{3}{2} - \frac{2}{k}$ for the remaining section and recall that for the 3 D - case we have chosen the auxiliary quantity as

$$|P_N u|_{(k)} := N^{s_c} \|I_x^{\frac{1}{10}} P_N u\|_{L_{xyt}^{\frac{15}{4}}} + N^{s_c} \|P_N u\|_{L_{xyt}^4} + N^{\frac{3}{4} - \frac{3}{2k}} \|P_N u\|_{L_{xy}^4 L_t^{6k}}.$$

By the linear estimates (30) and (31) the three contributions are controlled by

$$|P_N u|_{(k)} \lesssim N^{s_c} \|P_N u\|_{V_\phi^2}.$$

Again, if the time intervall is taken $(0, T)$ in the involved norms, we write $|P_N u|_{(k, T)}$ instead of $|P_N u|_{(k)}$ and then we can rely on $\lim_{T \rightarrow 0} |P_N u|_{(k)} = 0$ for all $u \in V_\phi^2$.

Lemma 5. *Let $u_1, \dots, u_{k+1} \in V_\phi^2$, $w \in U_\phi^2$ with $\|w\|_{U_\phi^2} \leq 1$, N, N_1, \dots, N_{k+1} dyadic numbers with $N_1 \leq N_2 \leq \dots \leq N_{k+1}$ and $N \lesssim N_{k+1}$. Then there exists $\varepsilon > 0$ such that*

$$N^{s_c} \left| \int_{\mathbb{R}^3} P_{N_1} u_1 \cdot \dots \cdot P_{N_{k+1}} u_{k+1} \cdot \partial_x P_N w dx dy dt \right| \lesssim N^\varepsilon N_1^\varepsilon N_{k+1}^{-2\varepsilon} \prod_{j=1}^{k+1} |P_{N_j} u_j|_{(k)}.$$

Proof. We consider two cases.

Case 1: $N_{k+1}^\varepsilon |\xi| \lesssim |\xi_{k+1}|^\varepsilon N$,

Case 2: $N_{k+1}^\varepsilon |\xi| \lesssim N_k^\varepsilon N$.

In fact there is no further alternative. Clearly, we are in Case 2, if $N_{k+1} \lesssim N_k$. Otherwise we have $N_{k+1} \lesssim N$ and hence $N_{k+1}^\varepsilon |\xi|^{1-\varepsilon} \lesssim N$. Now, since $\xi = \sum_{j=1}^{k+1} \xi_j$, we have

- (i) $|\xi| \lesssim |\xi_{k+1}|$, hence $|\xi|^\varepsilon \lesssim |\xi_{k+1}|^\varepsilon$ and we are in Case 1, or
- (ii) $|\xi| \lesssim |\xi_j|$ for one $j \in \{1, \dots, k\}$, hence $|\xi|^\varepsilon \lesssim N_k^\varepsilon$, and we are in Case 2 again.

Estimation for Case 1: We use $N_{k+1}^\varepsilon |\xi| \lesssim |\xi_{k+1}|^\varepsilon N$, $N \lesssim N_{k+1}$ and the Kato smoothing effect to obtain

$$(32) \quad N^\varepsilon N_{k+1}^{s_c-2\varepsilon} \|P_{N_1} u_1 \cdot \dots \cdot P_{N_k} u_k \cdot (I_x^\varepsilon P_{N_{k+1}} u_{k+1})\|_{L_x^1 L_{yt}^2}$$

as upper bound for the contribution from this case. We choose Hölder exponents p_j and q_j with

$$\frac{1}{p_1} = \frac{3}{4k} - \frac{\varepsilon}{6}, \quad \frac{1}{q_1} = \frac{1}{4k} - \frac{\varepsilon}{6},$$

and, for $j \in \{2, \dots, k\}$,

$$\frac{1}{p_j} = \frac{3}{4k}, \quad \frac{1}{q_j} = \frac{1}{4k},$$

as well as

$$\frac{1}{p_{k+1}} = \frac{1}{4} + \frac{\varepsilon}{6}, \quad \frac{1}{q_{k+1}} = \frac{1}{4} + \frac{\varepsilon}{6},$$

so that $\sum_{j=1}^{k+1} \frac{1}{p_j} = 1$ and $\sum_{j=1}^{k+1} \frac{1}{q_j} = \frac{1}{2}$. Hölder's inequality gives

$$\|P_{N_1} u_1 \cdot \dots \cdot P_{N_k} u_k \cdot (I_x^\varepsilon P_{N_{k+1}} u_{k+1})\|_{L_x^1 L_{yt}^2} \leq \left(\prod_{j=1}^k \|P_{N_j} u_j\|_{L_x^{p_j} L_{yt}^{q_j}} \right) \|I_x^\varepsilon P_{N_{k+1}} u_{k+1}\|_{L_{xyt}^{p_{k+1}}}.$$

For the first factor we use Sobolev embeddings in the space variables to obtain

$$\|P_{N_1} u_1\|_{L_x^{p_1} L_{yt}^{q_1}} = N_1^\varepsilon N_1^{-\varepsilon} \|P_{N_1} u_1\|_{L_x^{p_1} L_{yt}^{q_1}} \lesssim N_1^\varepsilon N_1^{\frac{3}{4} - \frac{5}{4k} - \frac{\varepsilon}{2}} \|P_{N_1} u_1\|_{L_{xyt}^4 L_t^{q_1}}.$$

Using a convexity inequality we can control $N_1^{\frac{3}{4} - \frac{5}{4k} - \frac{\varepsilon}{2}} \|P_{N_1} u_1\|_{L_{xyt}^4 L_t^{q_1}}$ by the second and third term in the auxiliary quantity $|\cdot|_{(k)}$ and we arrive at

$$\|P_{N_1} u_1\|_{L_x^{p_1} L_{yt}^{q_1}} \lesssim N_1^\varepsilon |P_{N_1} u_1|_{(k)}.$$

In this calculation for u_1 we may take $\varepsilon = 0$ and have for $j \in \{2, \dots, k\}$ the bound

$$\|P_{N_j} u_j\|_{L_x^{p_j} L_{yt}^{q_j}} \lesssim |P_{N_j} u_j|_{(k)}.$$

Finally for u_{k+1} we have $\|I_x^\varepsilon u\|_{L_{xyt}^{p_{k+1}}} \lesssim \|I_x^{\frac{1}{4}} u\|_{L_{xyt}^{\frac{15}{4}}} + \|u\|_{L_{xyt}^4}$ and hence

$$N_{k+1}^{s_c} \|I_x^\varepsilon P_{N_{k+1}} u_{k+1}\|_{L_{xyt}^{p_{k+1}}} \lesssim |P_{N_{k+1}} u_{k+1}|_{(k)}.$$

Summarizing we get

$$(32) \lesssim N^\varepsilon N_1^\varepsilon N_{k+1}^{-2\varepsilon} \prod_{j=1}^{k+1} |P_{N_j} u_j|_{(k)}.$$

Treatment of Case 2: Here we apply $|\xi| \lesssim N_k^\varepsilon N_{k+1}^{-\varepsilon} N$, eliminate the N by the application of the local smoothing estimate and remain with the task of estimating

$$(33) \quad N^\varepsilon N_k^\varepsilon N_{k+1}^{s_c-2\varepsilon} \|P_{N_1} u_1 \cdot \dots \cdot P_{N_{k+1}} u_{k+1}\|_{L_x^1 L_{yt}^2}.$$

We choose

$$\frac{1}{p_1} = \frac{3}{4k}, \quad \frac{1}{q_1} = \frac{1}{4k} - \frac{\varepsilon}{2}, \quad \frac{1}{r_1} = \frac{1}{4k},$$

for $j \in \{2, \dots, k-1\}$,

$$\frac{1}{p_j} = \frac{3}{4k}, \quad \frac{1}{q_j} = \frac{1}{4k}, \quad \frac{1}{r_j} = \frac{1}{4k},$$

as well as

$$\frac{1}{p_k} = \frac{3}{4k}, \quad \frac{1}{q_k} = \frac{1}{4k} + \frac{\varepsilon}{2}, \quad \frac{1}{r_k} = \frac{1}{4k},$$

and

$$\frac{1}{p_{k+1}} = \frac{1}{4}, \quad \frac{1}{q_{k+1}} = \frac{1}{4}, \quad \frac{1}{r_{k+1}} = \frac{1}{4}.$$

Hölder's inequality gives

$$\|P_{N_1} u_1 \cdots P_{N_{k+1}} u_{k+1}\|_{L_x^1 L_{yt}^2} \leq \left(\prod_{j=1}^k \|P_{N_j} u_j\|_{L_x^{p_j} L_y^{q_j} L_t^{r_j}} \right) \|P_{N_{k+1}} u_{k+1}\|_{L_{xyt}^4}.$$

Sobolev inequalities in x and y give

$$\|P_{N_1} u_1\|_{L_x^{p_1} L_y^{q_1} L_t^{r_1}} \lesssim N_1^\varepsilon N_1^{\frac{3}{4} - \frac{5}{4k}} \|P_{N_1} u_1\|_{L_{xy}^4 L_t^{4k}} \lesssim N_1^\varepsilon |P_{N_1} u_1|_{(k)},$$

the latter by earlier calculation. Similarly we have for $j \in \{2, \dots, k-1\}$ that $\|P_{N_j} u_j\|_{L_x^{p_j} L_y^{q_j} L_t^{r_j}} \lesssim |P_{N_j} u_j|_{(k)}$, and for the k th factor by almost the same Sobolev embeddings

$$\|P_{N_k} u_k\|_{L_x^{p_k} L_y^{q_k} L_t^{r_k}} \lesssim N_k^{-\varepsilon} N_k^{\frac{3}{4} - \frac{5}{4k}} \|P_{N_k} u_k\|_{L_{xy}^4 L_t^{4k}} \lesssim N_k^{-\varepsilon} |P_{N_k} u_k|_{(k)}.$$

The estimate for u_{k+1} is clear, since the L_{xyt}^4 - norm is a part of $|\cdot|_{(k)}$. Collecting terms we arrive at

$$(33) \lesssim N^\varepsilon N_1^\varepsilon N_{k+1}^{-2\varepsilon} \prod_{j=1}^{k+1} |P_{N_j} u_j|_{(k)},$$

which completes the calculation. \square

The further procedure is now the same as for the symmetrized equation in 2 D . From the quantities $|P_N u|_{(k)}$ and $|P_N u|_{(k,T)}$, respectively, one builds the auxiliary norms $\|u\|_{(k,q)}$ and $\|u\|_{(k,q,T)}$ as norms of Besov type. Since we avoided to use an L_t^∞ - norm, we have $\lim_{T \rightarrow 0} \|u\|_{(k,q,T)} = 0$, whenever u belongs to our solution space. For $u_1, \dots, u_{k+1} \in \dot{X}_q^{s_c}$ one defines

$$F(u_1, \dots, u_{k+1})(t) := \int_0^t U_\phi(t-s) \partial_x(u_1 \cdots u_{k+1})(s) ds.$$

Summation of the dyadic pieces as in Lemma 2 gives $F(u_1, \dots, u_{k+1}) \in \dot{X}_q^{s_c}$ and the estimate

$$\|F(u_1, \dots, u_{k+1})\|_{\dot{X}_q^{s_c}} \lesssim \prod_{j=1}^{k+1} \|u_j\|_{(k,q)},$$

which, if inserted into the proof of Theorem 2, leads to the claimed local and global well-posedness result in 3 D . No further argument comes in, which is specific for the 3 D - case.

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AXEL GRÜNROCK: HEINRICH-HEINE-UNIVERSITÄT DÜSSELDORF, MATHEMATISCHES INSTITUT,
UNIVERSITÄTSSTRASSE 1, 40225 DÜSSELDORF, GERMANY.

E-mail address: `gruenroc@math.uni-duesseldorf.de`